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A note on functional equations of prehomogeneous vector spaces of parabolic type arising from special linear Lie algebras

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Abstract

The purpose of the present note is twofold. Firstly, we review Shintani's calculation on the functional equation of prehomogeneous (local) zeta functions associated with the space of square matrices. Secondly, we show that Shintani's method can be applied to the prehomogeneous vector spaces of parabolic type arising from special linear Lie algebras, if we employ some recent results.

1 Introduction

The purpose of the present note is twofold. Firstly, we review Shintani's calculation ([7]) on the functional equation of prehomogeneous (local) zeta functions associated with the space of square matrices. Although this method can be applied to only a particular class of prehomogeneous vector spaces, when applied, the calculation is straightforward. Since the article [7] is written in Japanese and hardly available, it is worth recollecting Shintani's method here. Secondly, we show that Shintani's method can be applied to the prehomogeneous vector spaces of parabolic type arising from special linear Lie algebras, if we employ recent results of Amano ([1]) and the author ([10]).

Here let us recall the definition of prehomogeneous vector spaces of parabolic type (cf. Rubenthaler [4]). Let $\mathbb{G}$ be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$, and assume that $\mathfrak{g}$ has a grading $\mathfrak{g} = \bigoplus_k \mathfrak{g}_k$. Let $\mathbb{G}_0$ be the connected subgroup of $\mathbb{G}$ corresponding to the subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$. Via the adjoint representation, $\mathbb{G}_0$ acts on each $\mathfrak{g}_k$. Then it is known that $\mathfrak{g}_k$ $(k \neq 0)$ decomposes into a finite number of $\mathbb{G}_0$-orbits; in particular, $(\mathbb{G}_0, \mathfrak{g}_k)$ is a prehomogeneous vector space for $k \neq 0$. We note that there exist some complex semisimple Lie group $\mathbb{G}'$ with Lie algebra $\mathfrak{g}'$ and a grading $\mathfrak{g}' = \bigoplus_k (\mathfrak{g}')_k$ such that $(\mathbb{G}_0, \mathfrak{g}_k)$ is isomorphic

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to $((\mathbb{G}')_0,\ (\mathfrak{g}')_1)$. We call $(\mathbb{G}_0, \mathfrak{g}_1)$ a prehomogeneous vector space of parabolic type. Moreover, when $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, it is called a prehomogeneous vector space of *commutative* parabolic type. The space of square matrices, to which Shintani applied his method, is a prehomogeneous vector space of commutative parabolic type arising from special linear Lie algebra $\mathfrak{sl}_n$. The main point of this note is that the case of non commutative parabolic type can be treated in a similar way.

There are, however, two new ingredients. When Shintani's method is applied, one needs to calculate a certain Γ-integral. In the case of *irreducible* prehomogeneous vector spaces, Igusa [2] showed that under certain conditions, the Γ-integrals can be written explicitly in terms of b-functions. Igusa's result is generalized by Amano [1] to the *reducible* case, and Amano's result can be applied to the prehomogeneous vector spaces of our interest. Moreover, the author [10] has developed a method to calculate the b-functions for these prehomogeneous vector spaces. By giving an example, we demonstrate how Shintani's method can be combined with these recent results.

2 Review on Shintani's calculation

First we review Shintani's calculation on the local zeta functions associated with the space of square matrices. Let $G = GL_n(\mathbb{C}) \times SL_n(\mathbb{C}),\ V = M_n(\mathbb{C})$. The action of G on V is given by

$$g \cdot x = g_1 x g_2^{-1}$$

for $g = (g_1, g_2) \in G, x \in V$. Let $P(x) = \det x,\ \ S = \{x \in V\ ;\ P(x) = 0\}$. Then we have $V - S = G \cdot I_n$, and thus (G, V) is a regular prehomogeneous vector space. We identify V^* with V via $\langle x, y\rangle = \operatorname{tr} {}^t xy$. Then we have $P^* = P$. We denote by $G_{\mathbb{R}}^+$ the identity component of the real Lie group $G_{\mathbb{R}}$, and let

$$V_1 = \{x \in V_{\mathbb{R}}\ ;\ P(x) > 0\}, \qquad V_2 = \{x \in V_{\mathbb{R}}\ ;\ P(x) < 0\}.$$

Then, $V_{\mathbb{R}} - S_{\mathbb{R}} = V_1 \cup V_2$ is the $G_{\mathbb{R}}^+$-orbit decomposition. Let $b(s)$ be the b-function of $P(x) = \det x$, i.e., the polynomial of s satisfying

$$P\left(\frac{\partial}{\partial x}\right) P(x)^{s+1} = b(s) P(x)^s.$$

Then it is well known that $b(s) = \prod_{i=1}^n (s+i)$. We put $\gamma(s) = \prod_{i=1}^n \Gamma(s+i)$. Further, we denote by $\mathcal{S}(V_{\mathbb{R}})$ the space of rapidly decreasing functions on $V_{\mathbb{R}}$, and for $f \in \mathcal{S}(V_{\mathbb{R}})$, we define the Fourier transform $\widehat{f}$ of f by

$$\widehat{f}(y) = \int_{V_{\mathbb{R}}} f(x) \exp(2\pi\sqrt{-1}\,\langle x, y\rangle) dx.$$

Then, by [8], [9], we have an equality

$$\text{(2.1)} \quad \int_{V_i} |\det y|^{s-n} \cdot \widehat{f}(y)dy = \gamma(s-n) \sum_{j=1}^{2} (2\pi)^{-ns} \cdot e^{\frac{\pi\sqrt{-1}}{2}ns} \cdot \varepsilon_{ij}(s) t_{ij}(s) \int_{V_j} |\det x|^{-s} f(x)dx,$$

for $i = 1, 2$, where $\varepsilon_{ij}(s)$ $(i, j = 1, 2)$ is given by

$$\left(\varepsilon_{ij}(s)\right) = \begin{pmatrix} 1 & e^{-\pi\sqrt{-1}s} \\ e^{-\pi\sqrt{-1}s} & 1 \end{pmatrix},$$

and $t_{ij}(s)$ is a polynomial in $e^{-2\pi\sqrt{-1}s}$, for which no explicit formula is known. The equality (2.1) is called the *local functional equation* or the *Fundamental Theorem of prehomogeneous vector spaces*. In the following, we determine $t_{ij}(s)$ for this prehomogeneous vector space.

We put $\varepsilon_0 = \mathrm{diag}(-1, 1, \ldots, 1)$. Then we have

$$\text{(2.2)} \qquad \varepsilon_0 V_1 = V_2, \qquad \varepsilon_0 V_2 = V_1,$$

and for $f_{\varepsilon_0}(x) = f(\varepsilon_0 x)$, we have

$$\widehat{f_{\varepsilon_0}}(y) = \int_{V_{\mathbb{R}}} f(\varepsilon_0 x) e^{2\pi\sqrt{-1}\langle x, y\rangle} dx = \left(\widehat{f}\right)_{\varepsilon_0}(y).$$

Hence we have

$$\int_{V_1} |\det y|^s \widehat{f_{\varepsilon_0}}(y)dy = \int_{V_1} |\det y|^s \left(\widehat{f}\right)_{\varepsilon_0}(y)dy = \int_{V_2} |\det y|^s \widehat{f}(y)dy.$$

Similarly, we have

$$\int_{V_2} |\det y|^s \widehat{f_{\varepsilon_0}}(y)dy = \int_{V_2} |\det y|^s \left(\widehat{f}\right)_{\varepsilon_0}(y)dy = \int_{V_1} |\det y|^s \widehat{f}(y)dy.$$

Hence, if we put $c_{ij}(s) = (2\pi)^{-ns} \cdot e^{\frac{\pi\sqrt{-1}}{2}s} \cdot \varepsilon_{ij}(s) t_{ij}(s)$ and $C(s) = (c_{ij}(s))$, we have

$$C(s) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C(s),$$

and this implies $c_{11}(s) = c_{22}(s)$ and $c_{12}(s) = c_{21}(s)$. We thus obtain $t_{11}(s) = t_{22}(s)$ and $t_{12}(s) = t_{21}(s)$.

On the other hand, it is well known that

$$\text{(2.3)} \qquad \int_{M_n(\mathbb{R})} |\det x|^s e^{-\pi \operatorname{tr} {}^t x x} dx = \pi^{\frac{-ns}{2}} \prod_{j=1}^{n} \frac{\Gamma((s+j)/2)}{\Gamma(j/2)}.$$

By substituting $s \mapsto s - n$, we have

$$\int_{V_i} |\det y|^{s-n} e^{-\pi \operatorname{tr} {}^t y y} dy = \frac{1}{2} \cdot \pi^{\frac{n^2}{2} - \frac{sn}{2}} \cdot \prod_{j=0}^{n-1} \frac{\Gamma((s-j)/2)}{\Gamma((j+1)/2)}.$$

Since $\widehat{e^{-\pi \operatorname{tr} {}^tyy}} = e^{-\pi \operatorname{tr} {}^tyy}$, the above identity and the Fundamental Theorem (2.1) imply that

$$\pi^{\frac{n^2}{2}-\frac{ns}{2}} \prod_{j=0}^{n-1} \Gamma\left(\frac{s-j}{2}\right) = \pi^{\frac{ns}{2}} \prod_{j=0}^{n-1} \Gamma\left(\frac{-s+j+1}{2}\right) \gamma(s-n) \cdot (c_{11}(s) + c_{12}(s)).$$

Moreover, by using

$$\prod_{j=0}^{n-1} \Gamma\left(\frac{s-j}{2}\right) \cdot \gamma(s-n)^{-1} = 2^{n(1-s)} \cdot 2^{\frac{n(n-1)}{2}} \cdot \pi^{\frac{n}{2}} \cdot \prod_{j=0}^{n-1} \Gamma\left(\frac{s-j+1}{2}\right)^{-1},$$

$$\prod_{j=0}^{n-1} \Gamma\left(\frac{s-j+1}{2}\right)^{-1} \cdot \Gamma\left(\frac{-s+j+1}{2}\right)^{-1} = \prod_{j=0}^{n-1} \pi^{-1} \cdot \sin \pi\left(\frac{s-j+1}{2}\right),$$

we obtain

$$c_{11}(s) + c_{12}(s) = (2\pi)^{-ns} \cdot (2\pi)^{\frac{n(n-1)}{2}} \cdot 2^n \cdot \sin \frac{\pi(s+1)}{2} \cdots \sin \frac{\pi(s-n+2)}{2}.$$

This gives us the following relation among $t_{ij}(s)$:

$$e^{\frac{\pi\sqrt{-1}}{2}sn} \left(t_{11}(s) + e^{-\pi\sqrt{-1}s} t_{12}(s)\right) = (2\pi)^{\frac{n(n-1)}{2}} \cdot 2^n \cdot \sin \frac{\pi(s+1)}{2} \cdots \sin \frac{\pi(s-n+2)}{2}.$$

Recall that $t_{ij}(s)$ is a polynomial in $e^{-2\pi\sqrt{-1}s}$. Thus, if we substitute $s \mapsto s-1$ in the above identity, we obtain

$$e^{\frac{\pi\sqrt{-1}}{2}sn} \left(t_{11}(s) - e^{-\pi\sqrt{-1}s} t_{12}(s)\right) = (2\pi)^{\frac{n(n-1)}{2}} \cdot 2^n \cdot (\sqrt{-1})^n \sin \frac{\pi s}{2} \cdots \sin \frac{\pi(s-n+1)}{2}.$$

Solving two equations for $t_{ij}(s)$, we obtain

$$\begin{aligned} t_{11}(s) &= e^{-\frac{\pi\sqrt{-1}}{2}sn} \cdot (2\pi)^{\frac{n(n-1)}{2}} \cdot 2^{n-1} \left\{ \cos \frac{\pi s}{2} \cdots \cos \frac{\pi(s-n+1)}{2} \right. \\ &\qquad\qquad \left. + (\sqrt{-1})^n \sin \frac{\pi s}{2} \cdots \sin \frac{\pi(s-n+1)}{2} \right\} \\ &= t_{22}(s), \end{aligned}$$

$$\begin{aligned} t_{12}(s) &= e^{-\frac{\pi\sqrt{-1}}{2}s(n-2)} \cdot (2\pi)^{\frac{n(n-1)}{2}} \cdot 2^{n-1} \left\{ \cos \frac{\pi s}{2} \cdots \cos \frac{\pi(s-n+1)}{2} \right. \\ &\qquad\qquad \left. - (\sqrt{-1})^n \sin \frac{\pi s}{2} \cdots \sin \frac{\pi(s-n+1)}{2} \right\} \\ &= t_{21}(s). \end{aligned}$$

Remark. (1) The prehomogeneous vector space $(GL_n(\mathbb{C}) \times SL_n(\mathbb{C}), M_n(\mathbb{C}))$ can be considered as a prehomogeneous vector space of commutative parabolic type in the following way. Let $\mathbb{G} = SL_{2n}(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{C})$ and consider the grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $\mathfrak{g}$ defined by

$$\mathfrak{g} = \left(\begin{array}{c|c} \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right) \begin{array}{l} \}\, n \\ \}\, n \end{array}.$$

Then $(GL_n(\mathbb{C}) \times SL_n(\mathbb{C}), M_n(\mathbb{C}))$ is isomorphic to $(\mathbb{G}_0, \mathfrak{g}_1)$, where $\mathbb{G}_0$ is the connected subgroup of $\mathbb{G}$ corresponding to $\mathfrak{g}_0$.

(2) By the theory of prehomogeneous vector spaces, the functional equations of the (global) zeta functions

$$\xi(s) = \sum_{v \in SL_n(\mathbb{Z}) \backslash V_{\mathbb{Z}} \cap (V - S)} \frac{1}{|\det(v)|^s}$$

can be obtained from the local functional equations discussed in this section.

3 Igusa's result

Igusa [2] generalized the integral formula (2.3), which played a role in Shintani's calculation, to the prehomogeneous vector spaces satisfying the following conditions:

(A1) (G, ρ, V) is a reductive prehomogeneous vector space defined over $\mathbb{R}$.

(A2) (G, ρ, V) has a unique (up to constant) irreducible relative invariant $P(x)$, and the singular set S coincides with the zero set of $P(x)$: $S = \{x \in V\,;\, P(x) = 0\}$. Moreover, $V_{\mathbb{R}} - S_{\mathbb{R}}$ is a single $G_{\mathbb{R}}$-orbit.

(A3) $P(x)$ is multiplicity free, that is,

$$P(x) = \sum_{1 \le j_1 < \cdots < j_d \le n} a_{j_1, \ldots, j_d} x_{j_1} \cdots x_{j_d},$$

where $n = \dim V, d = \deg P$.

For simplicity, we further assume that $G \subset GL_N(\mathbb{R})$ and ${}^tG = G$. Then the b-function $b(s)$ of P is defined by

$$P\left(\frac{\partial}{\partial x}\right) P(x)^{s+1} = b(s) P(x)^s.$$

We normalize P so that the leading coefficient of $b(s)$ is equal to 1 and let

$$b(s) = \prod_{i=1}^{d} (s + \alpha_i).$$

Then Igusa proved the following formula for the Γ-integral:

$$\int_{V_{\mathbb{R}}} |P(x)|^s e^{-\pi \operatorname{tr} {}^t x x} dx = \pi^{-\frac{ds}{2}} \prod_{i=1}^{d} \frac{\Gamma(\frac{s+\alpha_i}{2})}{\Gamma(\frac{\alpha_i}{2})}.$$

We note that Igusa classified the irreducible prehomogeneous vector spaces satisfying the conditions (A1)–(A3), and Amano [1] generalized Igusa's integral formula to the case of multi-variables.

4 An example of calculation for the non commutative case

In this section, we give an example of calculation for the non commutative case. Let $n_2 > n_1 \geq 1$ and $N = 2n_1 + 2n_2$. We put $\mathbb{G} = SL_N(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C})$ and define a grading of $\mathfrak{g}$ as follows:

$$\mathfrak{g} = \begin{array}{c} \left(\begin{array}{c|c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 & \mathfrak{g}_3 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-3} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right) \end{array} \begin{array}{l} \}\, n_1 \\ \}\, n_2 \\ \}\, n_2 \\ \}\, n_1 \end{array}$$

Then we have

$$(\mathbb{G}_0, \mathfrak{g}_1) \cong (GL(n_1) \times GL(n_2) \times GL(n_2) \times SL(n_1), M(n_2, n_1) \oplus M(n_2, n_2) \oplus M(n_1, n_2)),$$

and the action is given as follows: for $g = (g_1, g_2, g_3, g_4) \in \mathbb{G}_0, v = (X_1, X_2, X_3) \in \mathfrak{g}_1$,

$$g \cdot v = \left(g_2 X_1 g_1^{-1},\, g_3 X_2 g_2^{-1},\, g_4 X_3 g_3^{-1}\right).$$

In the following, we write $G := \mathbb{G}_0, V := \mathfrak{g}_1$.

There exist two fundamental relative invariants $P_1(v)$ and $P_2(v)$, which is given by

$$P_1(v) = \det(X_3 X_2 X_1), \qquad P_2(v) = \det X_2,$$

with degrees $d_1 = 3n_1, d_2 = n_2$, respectively. We put $\underline{d} = (d_1, d_2) = (3n_1, n_2)$. Let $S = \{v \in V \ ; \ (P_1 P_2)(v) = 0\}$. Then $V - S$ is a single G-orbit, and thus (G, V) is a regular prehomogeneous vector space. Moreover, the $G_{\mathbb{R}}^{+}$-orbit decomposition of $V_{\mathbb{R}} - S_{\mathbb{R}}$ is given by

$$\begin{aligned} &V_{\mathbb{R}} - S_{\mathbb{R}} = V_1 \cup V_2 \cup V_3 \cup V_4, \\ &V_1 = \{v \in V_{\mathbb{R}} \,;\ \mathrm{sgn}\, P_1(v) = +,\ \mathrm{sgn}\, P_2(v) = +\}, \\ &V_2 = \{v \in V_{\mathbb{R}} \,;\ \mathrm{sgn}\, P_1(v) = +,\ \mathrm{sgn}\, P_2(v) = -\}, \\ &V_3 = \{v \in V_{\mathbb{R}} \,;\ \mathrm{sgn}\, P_1(v) = -,\ \mathrm{sgn}\, P_2(v) = +\}, \\ &V_4 = \{v \in V_{\mathbb{R}} \,;\ \mathrm{sgn}\, P_1(v) = -,\ \mathrm{sgn}\, P_2(v) = -\}. \end{aligned}$$

Let us remark here that it is not so easy to describe which partition (in this case, $N = n_1 + n_2 + n_2 + n_1$) makes $(\mathbb{G}_0, \mathfrak{g}_1)$ a *regular* prehomogeneous vector space; we refer to Mortajine [3] for this classification problem.

For a multi-variable $\underline{s} = (s_1, s_2)$, we write $P^{\underline{s}} = P^{s_1} P^{s_2}$, and so on. Then the "multivariable version" of the Fundamental Theorem of prehomogeneous vector spaces (cf. M. Sato–Shintani [7], F. Sato [5]) implies that

$$\int_{V_i^*} |P^*(v^*)|^{\underline{s}-\underline{\kappa}} \cdot \widehat{f}(v^*) dv^* = \gamma(\underline{s} - \underline{\kappa}) \sum_{j=1}^{l} c_{ij}(\underline{s}) \cdot \int_{V_j} |P(v)|^{-\underline{s}} f(v) dv \qquad (f \in \mathcal{S}(V_{\mathbb{R}})).$$

Here,

$$c_{ij}(\underline{s}) = (2\pi)^{-\underline{d}\cdot\underline{s}} \cdot \exp\left(\frac{\pi\sqrt{-1}}{2}\underline{d}\cdot\underline{s}\right)\varepsilon_{ij}(\underline{s})t_{ij}(\underline{s}),$$

$$\varepsilon_{ij}(\underline{s}) = \exp\left[-\frac{\pi\sqrt{-1}}{2}\sum_{\nu=1}^{r} s_\nu \cdot (1 - \epsilon_i^*(P_\nu^*)\epsilon_j(P_\nu))\right].$$

Moreover, $\gamma(\underline{s})$ is determined by the b-function, r is the number of the fundamental relative invariants, $\epsilon_i(*), \epsilon_j^*(*)$ are the characters with values ± 1 depending on the signature of the relative invariants, $t_{ij}(\underline{s})$ are polynomials in $\exp(-2\pi\sqrt{-1}s_1), \ldots, \exp(-2\pi\sqrt{-1}s_r)$.

In our case, we have $\underline{\kappa} = (n_2, n_2 - n_1)$, and the b-function $b_{\underline{m}}(\underline{s})$, which is defined by $P^{*\underline{m}}(\partial_v)P^{\underline{s}+\underline{m}}(v) = b_{\underline{m}}(\underline{s})P^{\underline{s}}(v)$, can be determined by using the decomposition formula for b-functions (F. Sato and Sugiyama [6]). The result is as follows:

$$\begin{aligned} b_{\underline{m}}(\underline{s}) = &\left\{\prod_{k=1}^{n_1}[s_1 + k]_{m_1} \times [s_1 + n_2 - n_1 + k]_{m_1}\right\} \\ &\times \left\{\prod_{k=1}^{n_2 - n_1}[s_2 + k]_{m_2}\right\} \times \left\{\prod_{k=1}^{n_1}[s_1 + s_2 + n_2 - n_1 + k]_{m_1 + m_2}\right\}. \end{aligned}$$

The symbol $[*]_m$ stands for $[A]_m = A(A+1)\cdots(A+m-1)$. We refer to Sugiyama [10] for the details. The above result on the b-function implies that

$$\begin{aligned} \gamma(\underline{s}) = &\left\{\prod_{k=1}^{n_1}\Gamma(s_1 + k)\Gamma(s_1 + n_2 - n_1 + k)\right\} \\ &\times \left\{\prod_{k=1}^{n_2 - n_1}\Gamma(s_2 + k)\right\} \times \left\{\prod_{k=1}^{n_1}\Gamma(s_1 + s_2 + n_2 - n_1 + k)\right\}. \end{aligned}$$

From the definition, $\varepsilon_{ij}(\underline{s})$ is given by

$$(\varepsilon_{ij}(\underline{s})) = \begin{pmatrix} 1 & e^{-\pi\sqrt{1}s_2} & e^{-\pi\sqrt{1}s_1} & e^{-\pi\sqrt{1}(s_1+s_2)} \\ e^{-\pi\sqrt{1}s_2} & 1 & e^{-\pi\sqrt{1}(s_1+s_2)} & e^{-\pi\sqrt{-1}s_1} \\ e^{-\pi\sqrt{-1}s_1} & e^{-\pi\sqrt{1}(s_1+s_2)} & 1 & e^{-\pi\sqrt{-1}s_2} \\ e^{-\pi\sqrt{1}(s_1+s_2)} & e^{-\pi\sqrt{-1}s_1} & e^{-\pi\sqrt{-1}s_2} & 1 \end{pmatrix}.$$

Now it remains to determine $t_{ij}(\underline{s})$. Since $V_{\mathbb{R}} - S_{\mathbb{R}}$ is a single $G_{\mathbb{R}}$-orbit, we can show that $(C(\underline{s})) := (c_{ij}(\underline{s}))$ satisfies

$$C(\underline{s})\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} C(\underline{s}),$$

$$C(\underline{s})\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} C(\underline{s}).$$

This implies that

$$t_{11}(\underline{s}) = t_{22}(\underline{s}) = t_{33}(\underline{s}) = t_{44}(\underline{s}), \qquad t_{12}(\underline{s}) = t_{21}(\underline{s}) = t_{34}(\underline{s}) = t_{43}(\underline{s}),$$
$$t_{13}(\underline{s}) = t_{24}(\underline{s}) = t_{31}(\underline{s}) = t_{42}(\underline{s}), \qquad t_{14}(\underline{s}) = t_{23}(\underline{s}) = t_{32}(\underline{s}) = t_{41}(\underline{s}),$$

and hence the calculation of $t_{ij}(\underline{s})$ is reduced to that of $t_{11}(\underline{s}), t_{12}(\underline{s}), t_{13}(\underline{s}),\ t_{14}(\underline{s})$. On the other hand, the result in Igusa [2] is generalized by Amano [1], and by using Amano's result, we have

$$\int_{V_{\mathbb{R}}} |P(v)|^{\underline{s}} \cdot e^{-\pi\,\mathrm{tr}({}^t vv)} dv = \pi^{-\frac{3}{2}n_1 s_1 - \frac{1}{2}n_2 s_2} \times \prod_{k=1}^{n_1} \frac{\Gamma(\frac{s_1+k}{2})}{\Gamma(\frac{k}{2})} \times \prod_{k=1}^{n_1} \frac{\Gamma(\frac{s_1+n_2-n_1+k}{2})}{\Gamma(\frac{n_2-n_1+k}{2})}$$
$$\times \prod_{k=1}^{n_2-n_1} \frac{\Gamma(\frac{s_2+k}{2})}{\Gamma(\frac{k}{2})} \times \prod_{k=1}^{n_1} \frac{\Gamma(\frac{s_1+s_2+n_2-n_1+k}{2})}{\Gamma(\frac{n_2-n_1+k}{2})}.$$

Then it follows that

$$t_{11}(\underline{s}) + e^{-\pi\sqrt{-1}s_2} t_{12}(\underline{s}) + e^{-\pi\sqrt{-1}s_1} t_{21}(\underline{s}) + e^{-\pi\sqrt{-1}(s_1+s_2)} t_{22}(\underline{s})$$
$$= e^{-\frac{\pi\sqrt{-1}}{2}(3n_1 s_1 + n_2 s_2)} \cdot 2^{2n_1+n_2} \cdot (2\pi)^{\frac{1}{2}(2n_1+n_2)(n_2-1)} \times \prod_{k=0}^{n_1-1} \sin\pi\left(\tfrac{s_1-k+1}{2}\right) \sin\pi\left(\tfrac{s_1-n_2+n_1-k+1}{2}\right)$$
$$\times \prod_{k=0}^{n_2-n_1-1} \sin\pi\left(\tfrac{s_2-k+1}{2}\right) \times \prod_{k=0}^{n_1-1} \sin\pi\left(\tfrac{s_1+s_2-n_2+n_1-k+1}{2}\right) \qquad \cdots\cdots (*1).$$

Recall that $t_{ij}(\underline{s})$ is a polynomial in $\exp(-2\pi\sqrt{-1}s_1), \exp(-2\pi\sqrt{-1}s_2)$. Then,

in $(*1)$, we change the variables $s_1 \mapsto s_1 - 1$, $s_2 \mapsto s_2$, then we obtain an equation $(*2)$,
in $(*1)$, we change the variables $s_1 \mapsto s_1$, $s_2 \mapsto s_2 - 1$, then we obtain an equation $(*3)$,
in $(*1)$, we change the variables $s_1 \mapsto s_1 - 1$, $s_2 \mapsto s_2 - 1$, then we obtain an equation $(*4)$.

Thus we have 4 independent equations $(*1)$, $(*2)$, $(*3)$, $(*4)$ for $t_{11}(\underline{s}), t_{12}(\underline{s}), t_{13}(\underline{s}),\ t_{14}(\underline{s})$, and hence we can determine explicitly $t_{11}(\underline{s}), t_{12}(\underline{s}), t_{13}(\underline{s}),\ t_{14}(\underline{s})$.

This method can be applied to (G, ρ, V) satisfying the following conditions:

(A1)' (G, ρ, V) is a reductive prehomogeneous vector space defined over $\mathbb{R}$,
(A2)' the singular set S is a hypersurface:

$$S = S_1 \cup \cdots \cup S_r, \qquad S_i = \{x \in V\,;\ P_i(x) = 0\},$$

and further, $V_{\mathbb{R}} - S_{\mathbb{R}}$ is a single $G_{\mathbb{R}}$-orbit, and each $G_{\mathbb{R}}^{+}$-orbit is characterized by the signatures of the relative invariants $P_i(x)$,
(A3)' each $P_i(x)$ is multiplicity free.

All the parabolic regular prehomogeneous vector spaces arising from the special linear Lie algebras satisfy the above conditions. Moreover, their b-functions can be calculated quite easily by using the result in [10]. This means that, if the space (i.e., partition) is given explicitly, we can calculate the Fourier transform in principle.

5 An application

Finally, we give an application to the global zeta functions. Let

$$\begin{aligned} G &= GL(1) \times GL(2) \times \cdots \times GL(n) \times SL(n) \times SL(n-1) \times \cdots \times SL(1), \\ V &= M(2,1) \oplus M(3,2) \oplus \cdots \oplus M(n,n-1) \\ &\qquad \oplus M(n,n) \\ &\qquad\qquad \oplus M(n-1,n) \oplus M(n-2,n-1) \oplus \cdots \oplus M(1,2). \end{aligned}$$

For

$$\begin{cases} \tilde{g} = (g_1, g_2, \ldots, g_n, h_n, h_{n-1}, \ldots, h_1) \in G, \qquad (g_i \in GL(i),\ h_i \in SL(i)) \\ v = (z_1, z_2, \ldots, z_{n-1}, X, y_{n-1}, y_{n-2}, \ldots, y_1) \in V, \\ \qquad\qquad (z_i \in M(i+1,i),\ X \in M(n,n),\ y_i \in M(i,i+1)) \end{cases},$$

we define

$$\tilde{g} \cdot v = \left(g_2 z_1 g_1^{-1}, \ldots, g_n z_{n-1} g_{n-1}^{-1},\ h_n X g_n^{-1},\ h_{n-1} y_{n-1} h_n^{-1}, \ldots, h_1 y_1 h_2^{-1}\right).$$

Note that (G,V) is a parabolic prehomogeneous vector space arising from the parabolic subalgebra corresponding to the partition $1+2+\cdots+n+n+\cdots+2+1$. However, we have adjusted the scalar multiplications so that the generic isotropy subgroup (G,V) becomes trivial. Moreover, there are n fundamental relative invariants:

$$\begin{aligned} P_1(v) &= \det(y_1 y_2 \cdots y_{n-1} X z_{n-1} \cdots z_2 z_1), \\ P_2(v) &= \det(y_2 \cdots y_{n-1} X z_{n-1} \cdots z_2), \\ &\ \vdots \qquad \vdots \\ P_{n-1}(v) &= \det(y_{n-1} X z_{n-1}), \\ P_n(v) &= \det X. \end{aligned}$$

The b-function $b_{\underline{m}}(\underline{s})$ is given by

$$\begin{aligned} b_{\underline{m}}(\underline{s}) = &\left\{ \prod_{1 \le i \le j \le n-1} [s_i + s_{i+1} + \cdots + s_j + j - i + 1]_{m_i + \cdots + m_j} \right. \\ &\qquad \left. [s_i + s_{i+1} + \cdots + s_j + j - i + 2]_{m_i + \cdots + m_j} \right\} \\ &\times \left\{ \prod_{i=1}^{n} [s_i + s_{i+1} + \cdots + s_n + n - i + 1]_{m_i + \cdots + m_n} \right\}. \end{aligned}$$

Now we define the zeta function by

$$\xi(s_1,\ldots,s_n)=\sum_{v\in G_{\mathbb{Z}}\backslash V_{\mathbb{Z}}\cap(V-S)}\frac{1}{|P_1(v)|^{s_1}\cdots|P_n(v)|^{s_n}}.$$

Then, by the theory of prehomogeneous vector spaces, we can show that $\xi(s_1,\ldots,s_n)$ satisfies the following functional equation.

$$\begin{aligned}
&\xi(2-s_1,2-s_2,\ldots,2-s_{n-1},1-s_n)\\
&=2^{n^2}\cdot(2\pi)^{-\sum_{i=1}^{n}i(2n-2i+1)s_i}\cdot(2\pi)^{\frac{1}{3}n(n-1)(n+1)}\\
&\quad\times\left\{\prod_{1\leq i\leq j\leq n-1}\Gamma(s_i+\cdots+s_j-j+i-1)\Gamma(s_i+\cdots+s_j-j+i)\right\}\\
&\quad\times\left\{\prod_{i=1}^{n}\Gamma(s_i+\cdots+s_n-n+i)\right\}\\
&\quad\times\left\{\prod_{1\leq i\leq j\leq n-1}\cos\pi\left(\frac{s_i+\cdots+s_j-j+i-1}{2}\right)\cos\pi\left(\frac{s_i+\cdots+s_j-j+i}{2}\right)\right\}\\
&\quad\times\left\{\prod_{i=1}^{n}\cos\pi\left(\frac{s_i+\cdots+s_n-n+i}{2}\right)\right\}\times\xi(s_1,s_2,\ldots,s_n).
\end{aligned}$$

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